

MULTIPLICITY AND ASYMPTOTIC PROFILE OF 2-NODAL SOLUTIONS TO A SEMILINEAR ELLIPTIC PROBLEM ON A RIEMANNIAN MANIFOLD

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ABSTRACT. We establish a lower bound for the number of sign changing solutions with precisely two nodal domains to the singularly perturbed nonlinear elliptic equation $-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u$ on an n -dimensional Riemannian manifold M , $p \in (2, 2^*)$, in terms of the cup-length of the configuration space of M . We give a precise description of the asymptotic profile of these solutions as $\varepsilon \rightarrow 0$. We also provide new estimates for the cup-length of the configuration space of M .

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1. INTRODUCTION

Let (M, g) be a compact connected Riemannian manifold, without boundary, of class \mathcal{C}^k with $k \geq 1$. Let $n \geq 2$ be the dimension of M . We consider the following problem

$$(1.1) \quad \begin{cases} -\varepsilon^2 \Delta_g u + u = |u|^{p-2}u, \\ u \in H_g^1(M), \end{cases}$$

with $2 < p < 2^*$, where $2^* := \frac{2n}{n-2}$ if $n > 2$ and $2^* = \infty$ if $n = 2$. The space $H_g^1(M)$ is the completion of $\mathcal{C}^\infty(M)$ with respect to the norm defined by $\|u\|_g^2 = \int_M (|\nabla_g u|^2 + u^2) d\mu_g$.

This problem resembles the Neumann problem on a flat domain, which has been widely studied in literature, see e.g. [7, 9, 12, 13, 16, 21, 22, 25, 26, 27].

Existence, multiplicity and shape of positive solutions to (1.1) have been investigated in several papers. The existence of a mountain pass solution was proved by Byeon and Park in [3], who also showed that this solution has a spike which approaches a maximum point of the scalar curvature as $\varepsilon \rightarrow 0$. Benci, Bonanno and Micheletti [2] showed that problem (1.1) has at least $\text{cat}(M) + 1$ positive solutions if ε is small enough, where $\text{cat}(M)$ denotes the Lusternik-Schnirelmann category of M . A similar result for more general nonlinearities was obtained in [24]. In [15] N. Hirano gave a lower bound for the number of positive solutions to (1.1) in terms of the category of a set determined by the geometry of M . Solutions with one peak

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which concentrates at a stable critical point of the scalar curvature of M as $\varepsilon \rightarrow 0$ were obtained by Micheletti and Pistoia in [19], and in [6] Dancer, Micheletti and Pistoia proved the existence solutions with k peaks which concentrate at an isolated minimum of the scalar curvature of M as $\varepsilon \rightarrow 0$.

Concerning sign changing solutions only few results are known. When the scalar curvature of M is not constant a solution with one positive peak and one negative peak, concentrating at a maximum and a minimum of the scalar curvature, was obtained by Micheletti and Pistoia in [18]. Multiplicity of solutions which change sign exactly once was established by Ghimenti and Micheletti in [11] for Riemannian manifolds M which are invariant with respect to an orthogonal involution.

Here we provide a lower bound for the number of sign changing solutions with precisely two nodal domains, without requiring any symmetry or geometric assumptions on the manifold M , and we give a precise description of the asymptotic profile of these solutions as $\varepsilon \rightarrow 0$.

In order to state our main result we introduce some notation. The limiting problem to problem (1.1) as $\varepsilon \rightarrow 0$ is

$$(1.2) \quad -\Delta u + u = |u|^{p-2}u, \quad u \in H^1(\mathbb{R}^n).$$

It is well known that, up to translations, this problem has a unique positive solution, which is spherically symmetric, and is usually denoted by $U \in H^1(\mathbb{R}^n)$.

The exponential map $\exp : TM \rightarrow M$, defined on the total space TM of the tangent bundle of M , is a \mathcal{C}^∞ map. Since M is compact, there exists $R > 0$ such that $\exp_q : B(0, R) \rightarrow B_g(q, R)$ is a diffeomorphism for every $q \in M$. Here $T_q M$ is identified with \mathbb{R}^n , $B(0, R)$ is the ball of radius R in \mathbb{R}^n centered at 0, and $B_g(q, R)$ denotes the ball of radius R in M centered at q with respect to the distance induced by the Riemannian metric g .

Let $\chi_R \in \mathcal{C}^\infty(\mathbb{R}^n)$ be a radial cut-off function such that $\chi_R(z) = 1$ if $|z| \leq R/2$, $\chi_R(z) = 0$ if $|z| \geq R$, and $|\nabla \chi_R(z)| < 2/R$ and $|\nabla^2 \chi_R(z)| < 2/R^2$ for all $z \in \mathbb{R}^n$. For $\xi \in M$ and $\varepsilon > 0$ we define $W_{\varepsilon, \xi} \in H_g^1(M)$ by

$$(1.3) \quad W_{\varepsilon, \xi}(x) = \begin{cases} U\left(\frac{\exp_\xi^{-1}(x)}{\varepsilon}\right) \chi_R\left(\frac{\exp_\xi^{-1}(x)}{\varepsilon}\right) & \text{if } x \in B_g(\xi, \varepsilon R), \\ 0 & \text{otherwise.} \end{cases}$$

Let $F(M) := \{(x, y) \in M \times M : x \neq y\}$. The quotient space $C(M)$ of $F(M)$ obtained by identifying (x, y) with (y, x) is called the configuration space of M .

We write \mathcal{H}^* for singular cohomology with coefficients in $\mathbb{Z}/2$. Recall that the cup-length of a topological space X is the smallest integer $k \geq 1$ such that the cup-product of any k cohomology classes in $\tilde{\mathcal{H}}^*(X)$ is zero, where $\tilde{\mathcal{H}}^*$ is reduced cohomology. We denote it by $\text{cupl} X$.

We are ready to state our main result.

Theorem 1.1. *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ problem (1.1) has at least $\text{cupl} C(M)$ pairs of sign changing solutions $\pm u_\varepsilon$ with the following properties:*

- (a) u_ε has a unique local maximum point Q_ε and a unique local minimum point q_ε on M .

(b) For any fixed $T > 0$,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \|u_\varepsilon(\exp_{Q_\varepsilon}(\varepsilon z)) - U(z)\|_{C^2(B(0,T))} &= 0, \\ \lim_{\varepsilon \rightarrow 0} \|u_\varepsilon(\exp_{q_\varepsilon}(\varepsilon z)) + U(z)\|_{C^2(B(0,T))} &= 0. \end{aligned}$$

(c) Moreover,

$$\begin{aligned} \sup_{\xi \in M \setminus B_g(Q_\varepsilon, \varepsilon T)} u_\varepsilon(\xi) &< ce^{-\mu T} + \sigma_1(\varepsilon), \\ \inf_{\xi \in M \setminus B_g(q_\varepsilon, \varepsilon T)} u_\varepsilon(\xi) &> -ce^{-\mu T} + \sigma_2(\varepsilon), \end{aligned}$$

for some positive constants c, μ and for some functions σ_1, σ_2 which go to zero as ε goes to zero.

(d) The function Φ_ε given by

$$u_\varepsilon = W_{\varepsilon, Q_\varepsilon} - W_{\varepsilon, q_\varepsilon} + \Phi_\varepsilon$$

is such that $\|\Phi_\varepsilon\|_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, with $\|\cdot\|_\varepsilon$ as in (2.1) below.

We shall see that

$$\text{cupl } C(M) \geq n + 1.$$

However, this estimate can be improved in many cases. We prove the following result.

Theorem 1.2. *If $\mathcal{H}^i(M) = 0$ for all $0 < i < m$ and if there are k cohomology classes $\zeta_1, \dots, \zeta_k \in \mathcal{H}^m(M)$ whose cup-product is nontrivial, then*

$$\text{cupl } C(M) \geq k + n.$$

For example, if M is homeomorphic to an n -torus $\mathbb{S}^1 \times \dots \times \mathbb{S}^1$ (with n factors) then $\text{cupl } C(M) = 2n$.

Unlike the case of positive solutions or the symmetric case considered in [11] where one looks for solutions on some Nehari manifold, there is no natural constraint for sign changing solutions to problem (1.1). Our approach is based on some ideas introduced in [1]. We exhibit a set \mathcal{Z}_ε of sign changing functions which is positively invariant under the negative gradient flow of the energy functional associated to problem (1.1). This set does not have an explicit description, so there is no way of defining a map from $C(M)$ into it. However, using Dold's fixed point transfer [10], one can obtain a homomorphism at the cohomological level. A careful study of the barycenter map introduced in [2] allows us to establish a lower bound for the equivariant Lusternik-Schnirelmann of low energy sublevel sets of \mathcal{Z}_ε and, hence, for the number of sign changing solutions.

In contrast to the Lyapunov-Schmidt reduction method, applied for example in [18], this method does not provide information on the asymptotic profile of the solutions obtained, as $\varepsilon \rightarrow 0$. We carry out a careful analysis in order to show that they have the properties described in Theorem 1.1.

Finally we wish to point out that, although configuration spaces have been widely studied, not much seems to be known about the multiplicative structure of their cohomology. So we believe that Theorem 1.2 has an interest of its own. A similar estimate for the cup-length of the configuration space of an open subset of \mathbb{R}^n has been given in [1].

The paper is organized as follows: In section 2 we discuss the variational setting for sign changing solutions to problem (1.1). In section 3 we prove the multiplicity statement in Theorem 1.1, and in section 4 we prove that the solutions have properties (a)-(d). Section 5 is devoted to the proof of Theorem 1.2.

2. THE VARIATIONAL SETTING

For $\varepsilon > 0$ we take

$$(2.1) \quad \begin{aligned} (u, v)_\varepsilon &:= \frac{1}{\varepsilon^n} \int_M (\varepsilon^2 \langle \nabla_g u, \nabla_g v \rangle + uv) d\mu_g, \\ \|u\|_\varepsilon^2 &:= \frac{1}{\varepsilon^n} \int_M (\varepsilon^2 |\nabla_g u|^2 + u^2) d\mu_g \end{aligned}$$

as the scalar product and the corresponding norm in $H_g^1(M)$. For any $u \in L_g^p(M)$ we put

$$|u|_{p,\varepsilon} := \left(\frac{1}{\varepsilon^n} \int_M |u|^p d\mu_g \right)^{\frac{1}{p}}, \quad |u|_{p,g} := \left(\int_M |u|^p d\mu_g \right)^{\frac{1}{p}},$$

and define

$$S_\varepsilon := \inf \left\{ \frac{\|u\|_\varepsilon^2}{|u|_{p,\varepsilon}^2} : u \in H_g^1(M), u \neq 0 \right\}.$$

The solutions of (1.1) are the critical points of the functional

$$J_\varepsilon : H_g^1(M) \rightarrow \mathbb{R}, \quad J_\varepsilon(u) = \frac{1}{2} \|u\|_\varepsilon^2 - \frac{1}{p} |u|_{p,\varepsilon}^p.$$

Any solution $u \neq 0$ of (1.1) lies on the Nehari manifold

$$\begin{aligned} \mathcal{N}_\varepsilon &:= \{u \in H_g^1(M) \setminus \{0\} : J'_\varepsilon(u)u = 0\} \\ &= \{u \in H_g^1(M) \setminus \{0\} : \|u\|_\varepsilon^2 = |u|_{p,\varepsilon}^p\}, \end{aligned}$$

which is a \mathcal{C}^2 -manifold diffeomorphic to the unit sphere in $H_g^1(M)$. Any sign changing solution of (1.1) belongs to the set

$$\mathcal{E}_\varepsilon := \{u \in H_g^1(M) : u^+, u^- \in \mathcal{N}_\varepsilon\} \subset \mathcal{N}_\varepsilon.$$

Here $u^+ := \max\{u, 0\}$ and $u^- := \min\{u, 0\}$. We set

$$c_\varepsilon := \inf_{\mathcal{N}_\varepsilon} J_\varepsilon.$$

It is easily checked that

$$c_\varepsilon = \frac{p-2}{2p} S_\varepsilon^{p/(p-2)}.$$

Analogously, we consider the functional

$$J_\infty : H^1(\mathbb{R}^n) \rightarrow \mathbb{R}, \quad J_\infty(v) = \frac{1}{2} \int_{\mathbb{R}^n} (|\nabla v|^2 + v^2) dx - \frac{1}{p} \int_{\mathbb{R}^n} |v|^p dx.$$

associated to the limit problem (1.2). Any solution $v \neq 0$ to this problem lies on the Nehari manifold

$$\mathcal{N}_\infty := \{u \in H^1(\mathbb{R}^n) \setminus \{0\} : J'_\infty(u)u = 0\}.$$

We set

$$c_\infty := \inf_{\mathcal{N}_\infty} J_\infty.$$

It is well known that there exists a unique positive solution $U \in H^1(\mathbb{R}^n)$ to the limit problem (1.2) which is spherically symmetric with respect to the origin, and that $J_\infty(U) = c_\infty$. Moreover,

$$(2.2) \quad \lim_{|x| \rightarrow \infty} U(x) |x|^{\frac{n-1}{2}} \exp |x| = b > 0.$$

Setting $U_\varepsilon(x) := U\left(\frac{x}{\varepsilon}\right)$ we have that

$$-\varepsilon^2 \Delta U_\varepsilon + U_\varepsilon = U_\varepsilon^{p-1}.$$

We recall the following property of the infima c_ε .

Lemma 2.1. $\lim_{\varepsilon \rightarrow 0} c_\varepsilon = c_\infty$.

Proof. See [2, 3]. □

We consider the negative gradient flow $\varphi_\varepsilon : \mathcal{G}_\varepsilon \rightarrow H_g^1(M)$ of J_ε defined by

$$\begin{cases} \frac{d}{dt} \varphi_\varepsilon(t, u) = -\nabla_\varepsilon J_\varepsilon(\varphi_\varepsilon(t, u)), \\ \varphi_\varepsilon(0, u) = u. \end{cases}$$

Here $\nabla_\varepsilon J_\varepsilon$ is the gradient of J_ε with respect to the scalar product $(\cdot, \cdot)_\varepsilon$ and $\mathcal{G}_\varepsilon = \{(t, u) : u \in H_g^1(M), 0 \leq t \leq T_\varepsilon(u)\}$, where $T_\varepsilon(u) \in (0, +\infty)$ is the maximal existence time for $\varphi_\varepsilon(t, u)$.

A subset \mathcal{D} of $H_g^1(M)$ is called strictly positively invariant for the flow φ_ε if $\varphi_\varepsilon(t, u) \in \text{int}(\mathcal{D})$ for all $u \in \mathcal{D}$ and $t \in (0, T_\varepsilon(u))$, where $\text{int}(\mathcal{D})$ denotes the interior of \mathcal{D} in $H_g^1(M)$. If \mathcal{D} is strictly positive invariant then the set

$$\mathcal{A}_\varepsilon(\mathcal{D}) := \{u \in H_g^1(M) : \varphi_\varepsilon(t, u) \in \mathcal{D} \text{ for some } t \in (0, T_\varepsilon(u))\}$$

is an open subset of $H_g^1(M)$.

Let $\mathcal{P} = \{u \in H_g^1(M) : u \geq 0\}$ be the convex cone of nonnegative functions and let

$$\mathcal{B}(\varepsilon, \pm \mathcal{P}) := \left\{ u \in H_g^1(M) : \text{dist}_\varepsilon(u, \pm \mathcal{P}) \leq \frac{1}{2} S_\varepsilon^{p/2(p-2)} \right\},$$

where $\text{dist}_\varepsilon(u, \mathcal{P}) := \min_{v \in \mathcal{P}} \|u - v\|_\varepsilon$, $\text{dist}_\varepsilon(u, -\mathcal{P}) := \min_{v \in -\mathcal{P}} \|u - v\|_\varepsilon$.

Lemma 2.2. *The following statements hold true:*

- (a) $(\mathcal{B}(\varepsilon, \mathcal{P}) \cup \mathcal{B}(\varepsilon, -\mathcal{P})) \cap \mathcal{E}_\varepsilon = \emptyset$. Hence, if $u \in \mathcal{B}(\varepsilon, \mathcal{P}) \cup \mathcal{B}(\varepsilon, -\mathcal{P})$ is a solution of (1.1) then u does not change sign.
- (b) $\mathcal{B}(\varepsilon, \pm \mathcal{P})$ is strictly positively invariant for the flow φ_ε .

Proof. The proof is analogous to that of Proposition 3.1 in [1]. We sketch it here for the reader's convenience. Note first that

$$(2.3) \quad |u^-|_{p, \varepsilon} = \min_{v \in \mathcal{P}} |u - v|_{p, \varepsilon} \leq S_\varepsilon^{-1/2} \min_{v \in \mathcal{P}} \|u - v\|_\varepsilon = S_\varepsilon^{-1/2} \text{dist}_\varepsilon(u, \mathcal{P}).$$

So, if $u \in \mathcal{E}_\varepsilon \cap \mathcal{B}(\varepsilon, \mathcal{P})$, then

$$0 < S_\varepsilon^{p/(p-2)} \leq \|u^-\|_\varepsilon^2 = |u^-|_{p, \varepsilon}^p \leq S_\varepsilon^{-p/2} \text{dist}_\varepsilon(u, \mathcal{P})^p \leq \frac{1}{2^p} S_\varepsilon^{p/(p-2)},$$

which is a contradiction. Hence, $\mathcal{E}_\varepsilon \cap \mathcal{B}(\varepsilon, \mathcal{P}) = \emptyset$. Similarly, $\mathcal{E}_\varepsilon \cap \mathcal{B}(\varepsilon, -\mathcal{P}) = \emptyset$. This proves (a).

Next, we prove assertion (b) for $\mathcal{B}(\varepsilon, \mathcal{P})$. A similar argument goes through for $\mathcal{B}(\varepsilon, -\mathcal{P})$.

The gradient $\nabla_\varepsilon J_\varepsilon$ with respect to the scalar product $(\cdot, \cdot)_\varepsilon$ is given by $\nabla_\varepsilon J_\varepsilon = Id - K_\varepsilon$, where $K_\varepsilon \in H_g^1(M)$ is defined by

$$(K_\varepsilon(u), v)_\varepsilon = \frac{1}{\varepsilon^n} \int_M |u|^{p-2} uv \, d\mu_g \quad \forall v \in H_g^1(M).$$

Using (2.3) we obtain

$$\begin{aligned} \text{dist}_\varepsilon(K_\varepsilon(u), \mathcal{P}) \|K_\varepsilon(u)^-\|_\varepsilon &\leq \|K_\varepsilon(u)^-\|_\varepsilon^2 = (K_\varepsilon(u), K_\varepsilon(u)^-)_\varepsilon = \frac{1}{\varepsilon^n} \int_M |u|^{p-2} u K_\varepsilon(u)^- \, d\mu_g \\ &= \frac{1}{\varepsilon^n} \int_M (u^+)^{p-1} K_\varepsilon(u)^- \, d\mu_g + \frac{1}{\varepsilon^n} \int_M |u^-|^{p-2} u^- K_\varepsilon(u)^- \, d\mu_g \\ &\leq \frac{1}{\varepsilon^n} \int_M |u^-|^{p-2} u^- K_\varepsilon(u)^- \, d\mu_g \leq |K_\varepsilon(u)^-|_{p,\varepsilon} |u^-|_{p,\varepsilon}^{p-1} \\ &\leq S_\varepsilon^{-1/2} \|K_\varepsilon(u)^-\|_\varepsilon S_\varepsilon^{-(p-1)/2} \text{dist}_\varepsilon(u, \mathcal{P})^{p-1}. \end{aligned}$$

Therefore, $\text{dist}_\varepsilon(K_\varepsilon(u), \mathcal{P}) \leq S_\varepsilon^{-p/2} \text{dist}_\varepsilon(u, \mathcal{P})^{p-1} \leq \frac{1}{2^{p-1}} S^{p/2(p-2)}$ if $u \in \mathcal{B}(\varepsilon, \mathcal{P})$. Hence $K_\varepsilon(u) \in \text{int}(\mathcal{B}(\varepsilon, \mathcal{P}))$. The rest of the argument is completely analogous to that of Proposition 3.1 of [1]. \square

For $\varepsilon > 0$ we set

$$\begin{aligned} \mathcal{D}_\varepsilon &:= \mathcal{B}(\varepsilon, \mathcal{P}) \cup \mathcal{B}(\varepsilon, -\mathcal{P}) \cup J_\varepsilon^0, \\ \mathcal{Z}_\varepsilon &:= H_g^1(M) \setminus \mathcal{A}_\varepsilon(\mathcal{D}_\varepsilon), \end{aligned}$$

where $J_\varepsilon^a := \{u \in H_g^1(M) : J_\varepsilon(u) \leq a\}$ for $a \in \mathbb{R}$. By Lemma 2.2 we have that \mathcal{D}_ε is strictly positively invariant for φ_ε . By Lemma 2.2 every function in \mathcal{Z}_ε is sign changing and every sign changing solution to problem (1.1) lies in \mathcal{Z}_ε . We set

$$d_\varepsilon := \inf_{\mathcal{Z}_\varepsilon} J_\varepsilon.$$

The following version of Ekeland's variational principle holds true in this setting.

Lemma 2.3. *Given $\varepsilon > 0$, $\delta > 0$ and $u \in \mathcal{Z}_\varepsilon$ such that $J_\varepsilon(u) \leq d_\varepsilon + \delta$, there exists $v \in \mathcal{Z}_\varepsilon$ such that $J_\varepsilon(v) \leq J_\varepsilon(u)$, $\|u - v\|_\varepsilon \leq \sqrt{\delta}$ and $\|\nabla_\varepsilon J_\varepsilon(v)\|_\varepsilon \leq \sqrt{\delta}$.*

Proof. The proof is completely analogous to that of Lemma 3.2 in [1]. \square

Proposition 2.4. *For every $\varepsilon > 0$ there exists $v_\varepsilon \in \mathcal{Z}_\varepsilon$ such that $J_\varepsilon(v_\varepsilon) = d_\varepsilon$ and v_ε is a sign changing solution of (1.1). Moreover $d_\varepsilon \geq 2c_\varepsilon$.*

Proof. If (u_k) is a minimizing sequence for J_ε in \mathcal{Z}_ε then, by Lemma 2.2, we may assume that $\|\nabla_\varepsilon J_\varepsilon(u_k)\|_\varepsilon \rightarrow 0$ as $k \rightarrow \infty$. Since M is compact, the embedding $H_g^1(M) \hookrightarrow L_g^p(M)$ is compact, cf. [14]. Therefore, $J_\varepsilon : H_g^1(M) \rightarrow \mathbb{R}$ satisfies the Palais-Smale condition. So, passing to a subsequence, we have that $u_k \rightarrow v_\varepsilon$ strongly in $H_g^1(M)$ and $J_\varepsilon(v_\varepsilon) = d_\varepsilon$. Since \mathcal{Z}_ε is closed in $H_g^1(M)$, $v_\varepsilon \in \mathcal{Z}_\varepsilon$ and, since \mathcal{Z}_ε is invariant under the negative gradient flow, v_ε is a stationary point for the flow, i.e. a solution of (1.1). Finally, note that every sign changing solution v of (1.1) satisfies that $v^\pm \in \mathcal{N}_\varepsilon$. So, by Lemma 2.1, $J_\varepsilon(v) \geq 2c_\varepsilon$. In particular, $J_\varepsilon(v_\varepsilon) = d_\varepsilon \geq 2c_\varepsilon$, as claimed. \square

For each $u \in H_g^1(M) \setminus \{0\}$ there exists a unique positive number $t_\varepsilon(u)$ such that $t_\varepsilon(u)u \in \mathcal{N}_\varepsilon$. This number is given by

$$(2.4) \quad t_\varepsilon^{p-2}(u) = \frac{\|u\|_\varepsilon^2}{|u|_{p,\varepsilon}^p}.$$

We consider the set

$$F_\varepsilon(M) := \{(q_1, q_2) \in M \times M : \text{dist}_g(q_1, q_2) \geq 2\varepsilon R\},$$

where dist_g is the distance in M , and define $\iota_\varepsilon : F_\varepsilon(M) \rightarrow H_g^1(M)$ by

$$(2.5) \quad \iota_\varepsilon(q_1, q_2) = t_\varepsilon(W_{\varepsilon, q_1})W_{\varepsilon, q_1} - t_\varepsilon(W_{\varepsilon, q_2})W_{\varepsilon, q_2},$$

where $W_{\varepsilon, q}$ is the function defined in (1.3).

Lemma 2.5. *For every $\varepsilon > 0$ the map $\iota_\varepsilon : F_\varepsilon(M) \rightarrow H_g^1(M)$ is continuous. For each $\delta > 0$ there exists $\varepsilon_0 > 0$ such that, if $\varepsilon < \varepsilon_0$ then*

$$\iota_\varepsilon(q_1, q_2) \in J_\varepsilon^{2c_\infty + \delta} \cap \mathcal{E}_\varepsilon \quad \text{for all } (q_1, q_2) \in (M \times M)_\varepsilon.$$

Proof. Since W_{ε, q_1} and W_{ε, q_2} have disjoint supports, $\iota_\varepsilon(q_1, q_2)^+ = t_\varepsilon(W_{\varepsilon, q_1})W_{\varepsilon, q_1} \in \mathcal{N}_\varepsilon$ and $\iota_\varepsilon(q_1, q_2)^- = -t_\varepsilon(W_{\varepsilon, q_2})W_{\varepsilon, q_2} \in \mathcal{N}_\varepsilon$. Therefore, $\iota_\varepsilon(q_1, q_2) \in \mathcal{E}_\varepsilon$ for every $(q_1, q_2) \in F_\varepsilon(M)$. Moreover,

$$J_\varepsilon(\iota_\varepsilon(q_1, q_2)) = J_\varepsilon(t_\varepsilon(W_{\varepsilon, q_1})W_{\varepsilon, q_1}) + J_\varepsilon(t_\varepsilon(W_{\varepsilon, q_2})W_{\varepsilon, q_2}).$$

The result now follows from Proposition 4.2 in [2]. \square

Proposition 2.6. $\lim_{\varepsilon \rightarrow 0} d_\varepsilon = 2c_\infty$.

Proof. Let $\delta > 0$. Arguing as in [4] we have that $\inf_{\mathcal{E}_\varepsilon} J_\varepsilon$ is attained and any minimizer of J_ε on \mathcal{E}_ε is a sign changing solution of (1.1). Since every sign changing solution lies in \mathcal{Z}_ε , from Proposition 2.4 and Lemma 2.5 we conclude that

$$(2.6) \quad 2c_\varepsilon \leq d_\varepsilon \leq \inf_{\mathcal{E}_\varepsilon} J_\varepsilon \leq 2c_\infty + \delta$$

for ε small enough. Passing to the limit as $\varepsilon \rightarrow 0$ and using Lemma 2.1 we conclude that $\lim_{\varepsilon \rightarrow 0} d_\varepsilon = 2c_\infty$, as claimed. \square

In the following sections we shall use the following result proved by Weth in [28].

Theorem 2.7. *There exists $\kappa_0 \in (0, c_\infty)$ such that $J_\infty(u) > 2c_\infty + \kappa_0$ for every sign changing solution u of the limit problem (1.2).*

3. MULTIPLICITY OF SOLUTIONS

The goal of this section is to prove the first assertion of Theorem 1.1. More precisely, we prove the following result.

Theorem 3.1. *There exists $\varepsilon_0 > 0$ such that for every $\varepsilon \in (0, \varepsilon_0)$ problem (1.1) has at least $\text{cupl } C(M)$ pairs of sign changing solutions $\pm u$ with $J(u) \leq d_\varepsilon + \kappa_0$.*

Here κ_0 is as in Theorem 2.7. We start with some lemmas.

Lemma 3.2. *Let $u_k \in \mathcal{Z}_{\varepsilon_k} \cap J_{\varepsilon_k}^{d_{\varepsilon_k} + \delta_k}$ where $\varepsilon_k, \delta_k > 0$ are such that $\varepsilon_k \rightarrow 0$ and $\delta_k \rightarrow 0$ as $k \rightarrow \infty$. Then*

$$\text{dist}_{\varepsilon_k}(u_k^\pm, \mathcal{N}_{\varepsilon_k}) \rightarrow 0 \quad \text{and} \quad J_{\varepsilon_k}(u_k^\pm) \rightarrow c_\infty \quad \text{as } k \rightarrow \infty.$$

Proof. By Lemma 2.3 we may assume without loss of generality that $\|\nabla_{\varepsilon_k} J_{\varepsilon_k}(u_k)\|_{\varepsilon_k} \rightarrow 0$. Since

$$\frac{p-2}{2p} \|u_k\|_{\varepsilon_k}^2 = J_{\varepsilon_k}(u_k) - \frac{1}{p} J'_{\varepsilon_k}(u_k)u_k \leq J_{\varepsilon_k}(u_k) + \|\nabla_{\varepsilon_k} J_{\varepsilon_k}(u_k)\|_{\varepsilon_k} \|u_k\|_{\varepsilon_k},$$

we have that $\|u_k\|_{\varepsilon_k}$ is uniformly bounded. Hence,

$$|J'_{\varepsilon_k}(u_k)u_k^\pm| = \left| \|u_k^\pm\|_{\varepsilon_k}^2 - |u_k^\pm|_{p,\varepsilon_k}^p \right| \rightarrow 0.$$

Consequently, the number $t_{\varepsilon_k}(u_k^\pm)$ defined by (2.4) tends to 1 and, therefore,

$$\text{dist}_{\varepsilon_k}(u_k^\pm, \mathcal{N}_{\varepsilon_k}) \leq \|u_k^\pm - t_{\varepsilon_k}(u_k^\pm)u_k^\pm\|_{\varepsilon_k} \rightarrow 0.$$

This, together with Lemma 2.1 and Proposition 2.6, implies that

$$\begin{aligned} 2c_\infty &\leq \lim_{k \rightarrow \infty} J_{\varepsilon_k}(t_{\varepsilon_k}(u_k^+)u_k^+) + \lim_{k \rightarrow \infty} J_{\varepsilon_k}(t_{\varepsilon_k}(u_k^-)u_k^-) \\ &= \lim_{k \rightarrow \infty} J_{\varepsilon_k}(u_k^+) + \lim_{k \rightarrow \infty} J_{\varepsilon_k}(u_k^-) = \lim_{k \rightarrow \infty} J_{\varepsilon_k}(u_k) = 2c_\infty. \end{aligned}$$

Therefore, $J_{\varepsilon_k}(u_k^\pm) \rightarrow c_\infty$. \square

Lemma 3.3. *For each $\eta \in (0, 1)$ there exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that, for every $u \in \mathcal{Z}_\varepsilon \cap J_{\varepsilon}^{d_\varepsilon + \delta}$ with $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0)$, we can find $q^{(1)} = q^{(1)}(u)$ and $q^{(2)} = q^{(2)}(u)$ in M such that*

$$(3.1) \quad \frac{1}{\varepsilon^n} \int_{B_g(q^{(1)}, \varepsilon R)} |u^+|^p d\mu_g > (1 - \eta) \frac{2p}{p-2} c_\infty$$

$$(3.2) \quad \frac{1}{\varepsilon^n} \int_{B_g(q^{(2)}, \varepsilon R)} |u^-|^p d\mu_g > (1 - \eta) \frac{2p}{p-2} c_\infty.$$

Proof. We prove (3.1). Arguing by contradiction we assume there exist $\eta \in (0, 1)$, $\varepsilon_k, \delta_k > 0$ and $u_k \in \mathcal{Z}_{\varepsilon_k} \cap J_{\varepsilon_k}^{d_{\varepsilon_k} + \delta_k}$ such that $\varepsilon_k \rightarrow 0$ and $\delta_k \rightarrow 0$ as $k \rightarrow \infty$ and

$$(3.3) \quad \frac{1}{\varepsilon_k^n} \int_{B_g(q, \varepsilon_k R)} |u_k^+|^p d\mu_g \leq (1 - \eta) \frac{2p}{p-2} c_\infty \quad \text{for all } q \in M.$$

Then, by Lemma 3.2,

$$(3.4) \quad \lim_{k \rightarrow +\infty} \|u_k^\pm\|_{\varepsilon_k}^2 = \lim_{k \rightarrow +\infty} |u_k^\pm|_{p,\varepsilon_k}^p = \frac{2p}{p-2} c_\infty.$$

We divide the proof into several steps.

Step 1. There exist $\vartheta > 0$, $T > 0$, $k_0 \in \mathbb{N}$ and, for each $k > k_0$, a point $q_k \in M$ such that

$$(3.5) \quad \frac{1}{\varepsilon_k^n} \int_{B_g(q_k, T\varepsilon_k)} |u_k^+|^p d\mu_g > \vartheta \quad \text{for all } k > k_0.$$

Proof of Step 1. For each k large enough we choose a finite partition $\{M_j^k : j \in \Lambda_k\}$ of M such that M_j^k is closed, $M_i^k \cap M_j^k \subset \partial M_i^k \cap \partial M_j^k$ for any $i \neq j$, and there exist $T > 1$ and points $q_j^k \in M_j^k$ satisfying

$$B_g(q_j^k, \varepsilon_k) \subset M_j^k \subset B_g(q_j^k, T\varepsilon_k) \quad \text{for all } j \in \Lambda_k$$

and such that each point $x \in M$ is contained in at most m balls $B_g(q_j^k, T\varepsilon_k)$, where the number m does not depend on k . We denote by $u_{k,j}^+$ the restriction of u_k^+ to the set M_j^k . Then

$$(3.6) \quad |u_k^+|_{p,\varepsilon_k}^p = \sum_j |u_{k,j}^+|_{p,\varepsilon_k}^p \leq \max_j |u_{k,j}^+|_{p,\varepsilon_k}^{p-2} \sum_j |u_{k,j}^+|^2_{p,\varepsilon_k}.$$

We take a smooth cut-off function χ_k such that $\chi_k(t) \equiv 1$ if $0 < t < \varepsilon_k$ and $\chi_\varepsilon(t) \equiv 0$ if $t > T\varepsilon_k$ and $|\chi'_\varepsilon| \leq \frac{1}{\alpha\varepsilon_k}$. Then

$$\tilde{u}_{k,j}(x) := u_k^+(x)\chi_k(|x - q_j^k|) \in H_g^1(M).$$

Hence,

$$\begin{aligned} \left| u_{k,j}^+ \right|_{p,\varepsilon_k}^2 &\leq \left| \tilde{u}_{k,j} \right|_{p,\varepsilon_k}^2 \leq c \left\| \tilde{u}_{k,j} \right\|_{\varepsilon_k}^2 \\ &= c \left\| u_{k,j}^+ \right\|_{\varepsilon_k}^2 + c \left\| \tilde{u}_{k,j} \right\|_{B_g(q_j^k, T\varepsilon_k) \setminus M_j^k}^2 \\ &\leq c \left\| u_{k,j}^+ \right\|_{\varepsilon_k}^2 + c \left(\frac{2}{\alpha^2} + 1 \right) \left\| u_k^+ \right\|_{B_g(q_j^k, T\varepsilon_k) \setminus M_j^k}^2 \end{aligned}$$

and, therefore,

$$(3.7) \quad \sum_j \left| u_{k,j}^+ \right|_{p,\varepsilon_k}^2 \leq c \left\| u_k^+ \right\|_{\varepsilon_k}^2 + c \left(\frac{2}{\alpha^2} + 1 \right) m \left\| u_k^+ \right\|_{\varepsilon_k}^2.$$

Combining (3.6) and (3.7) we obtain

$$(3.8) \quad \left| u_k^+ \right|_{p,\varepsilon_k}^p \leq c \left\| u_k^+ \right\|_{\varepsilon_k}^2 \max_j \left| u_{k,j}^+ \right|_{p,\varepsilon_k}^{p-2}.$$

which together with (3.4) implies that there exist $\vartheta > 0$ and, for each k large enough, a $j \in \Lambda_k$ such that

$$\vartheta < \left| u_{k,j}^+ \right|_{p,\varepsilon_k}^p = \frac{1}{\varepsilon_k^n} \int_{M_j^k} |u_k^+|^p d\mu_g \leq \frac{1}{\varepsilon_k^n} \int_{B_g(q_j^k, T\varepsilon_k)} |u_k^+|^p d\mu_g.$$

This proves Step 1. \square

Since M is compact, we may assume that the sequence (q_k) converges to a point $q \in M$. We define $w_k \in H^1(\mathbb{R}^n)$ by

$$(3.9) \quad w_k(z) := \chi(\varepsilon_k |z|) u_k(\exp_{q_k}(\varepsilon_k z)),$$

where χ is a smooth cut-off function such that $\chi(t) \equiv 1$ for $0 \leq t \leq \frac{R}{2}$, $\chi(t) \equiv 0$ for $t \geq R$ and $|\chi'(t)| \leq \frac{2}{R}$. By (3.4) the sequence $(\|u_k\|_{\varepsilon_k})$ is bounded. Therefore, (w_k) is bounded in $H^1(\mathbb{R}^n)$ (see Lemma 5.6 in [2]). So, up to a subsequence, $w_k \rightharpoonup w$ weakly in $H^1(\mathbb{R}^n)$, $w_k \rightarrow w$ a.e. in \mathbb{R}^n and $w_k \rightarrow w$ strongly in $L_{loc}^p(\mathbb{R}^n)$. The following statement holds true.

Step 2. $w \in H^1(\mathbb{R}^n)$ is a weak solution of the equation $-\Delta w + w = |w|^{p-2}w$ and $w^+ \not\equiv 0$.

Proof of Step 2. Following the argument in the proof of Lemma 5.7 in [2] one shows that w is a weak solution of $-\Delta w + w = |w|^{p-2}w$. Next we show that $w^+ \not\equiv 0$. By Step 1 we have that, for k large,

$$\begin{aligned} |w_k^+|_{L^p(B(0,T))}^p &= \frac{1}{\varepsilon_k^n} \int_{B(0,\varepsilon_k T)} |u_k^+(\exp_{q_k}(y))|^p dy \\ &\geq \frac{1}{2\varepsilon_k^n} \int_{B(0,\varepsilon_k T)} |u_k^+(\exp_{q_k}(y))|^p |g_{q_k}(y)|^{1/2} dy \\ &= \frac{1}{2\varepsilon_k^n} \int_{B_g(q_k, \varepsilon_k T)} |u_k^+(x)|^p d\mu_g \geq \frac{\vartheta}{2}. \end{aligned}$$

Here we use the fact that $\lim_{k \rightarrow +\infty} |g_{q_k}(y)|^{1/2} = |g_q(0)|^{1/2} = 1$ because $q_k \rightarrow q$ and $|y| \leq \varepsilon_k T$, hence $|g_{q_k}(y)|^{1/2} \geq \frac{1}{2}$ for k large. Since $w_k \rightarrow w$ strongly in $L^p_{loc}(\mathbb{R}^n)$, passing to the limit we obtain that

$$|w^+|_{L^p(B(0,T))}^p = \lim_{k \rightarrow +\infty} |w_k^+|_{L^p(B(0,T))}^p \geq \frac{\vartheta}{2} > 0.$$

This proves Step 2. \square

Step 3. $J_\infty(w) = c_\infty$ and $w > 0$.

Proof of Step 3. Fix $\tau > 0$. Since $|g_{q_k}(\varepsilon_k z)|$ converges to $|g_q(0)| = 1$ then, for any $a \in (0, 1)$, one has that $|g_{q_k}(\varepsilon_k z)|^{-1/2} \leq (1-a)^{-1}$ for k large enough and $z \in B(0, \tau)$. Therefore, for k large enough we have

$$\begin{aligned} \int_{B(0,\tau)} |w_k(z)|^p dz &= \int_{B(0,\tau)} \chi^p(\varepsilon_k z) |u_k(\exp_{q_k}(\varepsilon_k z))|^p dz \\ &\leq \left(\frac{1}{1-a} \right) \frac{1}{\varepsilon_k^n} \int_{B(0,\tau\varepsilon_k)} |u_k(\exp_{q_k}(y))|^p |g_{q_k}(y)|^{\frac{1}{2}} dy \\ (3.10) \quad &\leq \left(\frac{1}{1-a} \right) \frac{1}{\varepsilon_k^n} \int_{B_g(q_k, \tau\varepsilon_k)} |u_k(x)|^p d\mu_g \leq \frac{1}{1-a} |u_k|_{p, \varepsilon_k}^p. \end{aligned}$$

Then, by (3.4), we have

$$\int_{B(0,\tau)} |w(z)|^p dz = \lim_{k \rightarrow +\infty} \int_{B(0,\tau)} |w_k(z)|^p dz \leq \frac{2p}{p-2} 2c_\infty.$$

Hence,

$$\int_{\mathbb{R}^n} |w(z)|^p dz = \lim_{\tau \rightarrow +\infty} \int_{B(0,\tau)} |w_k(z)|^p dz \leq \frac{2p}{p-2} 2c_\infty.$$

It follows that $J_\infty(w) = \frac{p-2}{2p} |w|_{L^p(\mathbb{R}^n)}^p \leq 2c_\infty$. Theorem 2.7 implies that $w > 0$ and $J_\infty(w) = c_\infty$. This proves Step 3. \square

We are now ready to prove Lemma 3.3.

Fix $a \in (0, \eta)$. Since w_k^+ converges to $w = w^+$ in $L^p_{loc}(\mathbb{R}^n)$ and $J_\infty(w) = c_\infty$, there exists $\tau > 0$ such that, for k large enough,

$$\left(\frac{1-\eta}{1-a} \right) \frac{2p}{p-2} c_\infty < \int_{B(0,\tau)} |w_k^+(z)|^p dz.$$

On the other hand, arguing as in Step 3 and using (3.3) we have that, for k large enough,

$$\begin{aligned} \int_{B(0,\tau)} |w_k^+(z)|^p dz &\leq \left(\frac{1}{1-a} \right) \frac{1}{\varepsilon_k^n} \int_{B(0,\tau\varepsilon_k)} |u_k^+(\exp_{q_k}(y))|^p |g_{q_k}(y)|^{\frac{1}{2}} dy \\ &= \left(\frac{1}{1-a} \right) \frac{1}{\varepsilon_k^n} \int_{B_g(q_k, \varepsilon_k \tau)} |u_k^+(x)|^p d\mu_g \leq \left(\frac{1-\eta}{1-a} \right) \frac{2p}{p-2} c_\infty. \end{aligned}$$

This is a contradiction.

The proof of (3.2) is similar. This concludes the proof of Lemma 3.3. \square

By Nash's embedding theorem [20] we may assume that M is isometrically embedded in some euclidean space \mathbb{R}^N . We fix $r > 0$ such that $V_r := \{x \in \mathbb{R}^N : \text{dist}(x, M) \leq r\}$ is a tubular neighborhood of M . For $x \in V_r$ let $\pi(x) \in M$ be the unique point in M such that $|x - \pi(x)| = \text{dist}(x, M) \leq r$. The map $\pi : V_r \rightarrow M$ is

smooth and it is the normal disk bundle of the embedding $M \hookrightarrow \mathbb{R}^N$. We consider the map $\beta : \mathcal{N}_\varepsilon \rightarrow \mathbb{R}^N$ given by

$$\beta(u) := \frac{\int_M x |u(x)|^p d\mu_g}{\int_M |u(x)|^p d\mu_g}.$$

If $\beta(u) \in V_r$ we set

$$\beta_M(u) := \pi(\beta(u)).$$

The following statement holds true.

Proposition 3.4. *There exist $\delta_0 > 0$ and $\varepsilon_0 > 0$ such that, for every $u \in \mathcal{Z}_\varepsilon \cap J_\varepsilon^{d_\varepsilon + \delta}$ with $\varepsilon \in (0, \varepsilon_0)$ and $\delta \in (0, \delta_0]$,*

$$(3.11) \quad \beta(u^+) \in V_r, \quad \beta(u^-) \in V_r \quad \text{and} \quad \beta_M(u^+) \neq \beta_M(u^-).$$

Proof. Arguing by contradiction, let $\varepsilon_k, \delta_k > 0$ and $u_k \in \mathcal{Z}_{\varepsilon_k} \cap J_{\varepsilon_k}^{d_{\varepsilon_k} + \delta_k}$ be such that $\varepsilon_k \rightarrow 0$, $\delta_k \rightarrow 0$ and, for each k , one of the three statements in (3.11) is not true.

By Lemma 2.3 there exist $v_k \in \mathcal{Z}_{\varepsilon_k} \cap J_{\varepsilon_k}^{d_{\varepsilon_k} + \delta_k}$ such that $\|u_k - v_k\|_{\varepsilon_k} \leq \sqrt{\delta_k}$ and $\|\nabla_{\varepsilon_k} J_{\varepsilon_k}(v)\|_{\varepsilon_k} \leq \sqrt{\delta_k}$. Then, an easy estimate shows that

$$(3.12) \quad |\beta(u_k^\pm) - \beta(v_k^\pm)| \rightarrow 0.$$

By Lemma 3.3, after passing to a subsequence, there exist $q_k^1, q_k^2 \in M$ such that

$$(3.13) \quad \begin{aligned} \frac{2p}{p-2}(1 - \frac{1}{k})c_\infty &< \frac{1}{\varepsilon_k^n} \int_{B_g(q_k^1, \varepsilon_k R)} |v_k^+|^p d\mu_g \\ &\leq \frac{1}{\varepsilon_k^n} \int_M |v_k^+|^p d\mu_g \leq \frac{2p}{p-2}(1 + \frac{1}{k})c_\infty \end{aligned}$$

and

$$(3.14) \quad \begin{aligned} \frac{2p}{p-2}(1 - \frac{1}{k})c_\infty &< \frac{1}{\varepsilon_k^n} \int_{B_g(q_k^2, \varepsilon_k R)} |v_k^-|^p d\mu_g \\ &\leq \frac{1}{\varepsilon_k^n} \int_M |v_k^-|^p d\mu_g \leq \frac{2p}{p-2}(1 + \frac{1}{k})c_\infty. \end{aligned}$$

To simplify notation we write $\rho(v) := \frac{|v|^p}{\int_M |v|^p d\mu_g}$. Then we have

$$\begin{aligned} |\beta(v_k^+) - q_k^1| &= \left| \int_M (x - q_k^1) \rho(v_k^+) d\mu_g \right| \\ &\leq \left| \int_{B_g(q_k^1, \varepsilon_k R)} (x - q_k^1) \rho(v_k^+) d\mu_g \right| \\ &\quad + \left| \int_{M \setminus B_g(q_k^1, \varepsilon_k R)} (x - q_k^1) \rho(v_k^+) d\mu_g \right| \\ &\leq \varepsilon_k R + \frac{c}{k} \end{aligned}$$

for some positive constant c . This inequality, together with (3.12), implies that $\beta(u_k^+) \in V_r$ for k large enough. Similarly, $|\beta(u_k^-) - q_k^2| \rightarrow 0$ and $\beta(u_k^-) \in V_r$ for k large enough. Therefore u_k must satisfy $\beta_M(u_k^+) = \beta_M(u_k^-)$.

Since M is compact, after passing to a subsequence, we have that $q_k^1 \rightarrow q$ and $q_k^2 \rightarrow q$. The limit is the same because $\beta_M(u_k^+) = \beta_M(u_k^-)$.

Let χ be a smooth cut-off function such that $\chi(z) \equiv 1$ for $|z| \leq \frac{T}{2}$ and $\chi(z) \equiv 0$ for $|z| \geq T$, $T \in (0, R)$. Set

$$w_k^i(z) := \chi(\varepsilon_k z) v_k(\exp_{q_k^i}(\varepsilon_k z)), \quad i = 1, 2.$$

Then, up to a subsequence, $w_k^i \rightharpoonup w^i$ weakly in $H^1(\mathbb{R}^n)$, $w_k^i \rightarrow w^i$ a.e. in \mathbb{R}^n and $w_k^i \rightarrow w^i$ strongly in $L_{loc}^p(\mathbb{R}^n)$. Arguing as in the proof of Lemma 3.3 we conclude that w^i solves $-\Delta w + w = |w|^{p-2}w$ and $J_\infty(w^i) \leq 2c_\infty$. Inequality (3.13) implies that $(w^1)^+ \neq 0$. Hence, $w^1 > 0$. Similarly, $w^2 < 0$.

Next, we consider

$$w_k(z) := \chi(\varepsilon_k z) v_k(\exp_q(\varepsilon_k z)).$$

Again, up to a subsequence, $w_k \rightharpoonup w$ weakly in $H^1(\mathbb{R}^n)$, $w_k \rightarrow w$ a.e. in \mathbb{R}^n and $w_k \rightarrow w$ strongly in $L_{loc}^p(\mathbb{R}^n)$. For every $\varphi \in C_c^\infty(\mathbb{R}^n)$ and k large enough we have that

$$\begin{aligned} \int_{\mathbb{R}^n} w_k(z) \varphi(z) dz &= \int_{\mathbb{R}^n} w_k^1(\psi_k(z)) \varphi(z) dz \\ &= \int_{\mathbb{R}^n} w_k^1(y) \varphi(\psi_k^{-1}(z)) |\det \psi_k'(z)| dz, \end{aligned}$$

where $\psi_k(z) := \varepsilon_k^{-1} \exp_{q_k^1}^{-1}(\exp_q(\varepsilon_k z))$. Passing to the limit as $k \rightarrow \infty$ we obtain that

$$\int_{\mathbb{R}^n} w(z) \varphi(z) dz = \int_{\mathbb{R}^n} w^1(z) \varphi(z) dz \quad \text{for every } \varphi \in C_c^\infty(\mathbb{R}^n).$$

Hence, $w = w^1 > 0$. Similarly, we conclude that $w = w^2 < 0$. This is a contradiction. \square

Fix $\delta_0 > 0$ and $\varepsilon_0 > 0$ as in Proposition 3.4 and such that the map $\iota_\varepsilon : F_\varepsilon(M) \rightarrow J_\varepsilon^{d_\varepsilon + \delta_0} \cap \mathcal{E}_\varepsilon$ given by (2.5) is well defined for $\varepsilon \in (0, \varepsilon_0)$, where

$$F_\varepsilon(M) := \{(x, y) \in M \times M : \text{dist}_g(x, y) \geq 2\varepsilon R\} \subset F(M).$$

Define $\theta_\varepsilon : \mathcal{Z}_\varepsilon \cap J_\varepsilon^{d_\varepsilon + \delta_0} \rightarrow F(M)$ by

$$\theta_\varepsilon(u) := (\beta_M(u^+), \beta_M(u^-)).$$

The group $\mathbb{Z}/2 := \{1, -1\}$ acts on $F(M)$ by $-1 \cdot (x, y) := (y, x)$. Its $\mathbb{Z}/2$ -orbit space is the configuration space $C(M)$. Similarly, we denote by $C_\varepsilon(M)$ the $\mathbb{Z}/2$ -orbit space of $F_\varepsilon(M)$. On the other hand, $\mathbb{Z}/2$ acts on \mathcal{E}_ε and \mathcal{Z}_ε by multiplication. These actions are free, and the maps ι_ε and θ_ε are $\mathbb{Z}/2$ -equivariant, i.e.

$$\iota_\varepsilon(x, y) = -\iota_\varepsilon(y, x) \quad \text{and} \quad \theta_\varepsilon(-u) = -1 \cdot \theta_\varepsilon(u).$$

Hence, they induce maps between the $\mathbb{Z}/2$ -orbit spaces. We write $(\mathcal{Z}_\varepsilon \cap J_\varepsilon^{d_\varepsilon + \delta_0}) / (\mathbb{Z}/2)$ for the $\mathbb{Z}/2$ -orbit space of $\mathcal{Z}_\varepsilon \cap J_\varepsilon^{d_\varepsilon + \delta_0}$ and put a hat over a function to denote the induced map between $\mathbb{Z}/2$ -orbit spaces, e.g. we write

$$\widehat{\theta}_\varepsilon : (\mathcal{Z}_\varepsilon \cap J_\varepsilon^{d_\varepsilon + \delta_0}) / (\mathbb{Z}/2) \rightarrow C(M)$$

for the map induced by θ_ε . Let $\check{\mathcal{H}}$ be Alexander-Spanier or Čech cohomology with $\mathbb{Z}/2$ -coefficients. Recall that it coincides with singular cohomology \mathcal{H}^* on manifolds or, more generally, on ENRs. The following statement holds true.

Proposition 3.5. *There exists a homomorphism*

$$\tau_\varepsilon : \check{\mathcal{H}}((\mathcal{Z}_\varepsilon \cap J_\varepsilon^{d_\varepsilon + \delta_0}) / (\mathbb{Z}/2)) \rightarrow \mathcal{H}^*(C_\varepsilon(M))$$

such that the composition

$$\tau_\varepsilon \circ \hat{\theta}_\varepsilon^* : \mathcal{H}^*(C(M)) \rightarrow \mathcal{H}^*(C_\varepsilon(M))$$

is the homomorphism induced by the inclusion $C_\varepsilon(M) \hookrightarrow C(M)$, which is an isomorphism for ε small enough.

Proof. The proof is analogous to that of Proposition 5.2 in [1]. The idea is to express a certain subset of $\mathcal{Z}_\varepsilon \cap J_\varepsilon^{d_\varepsilon + \delta_0}$ as a fixed point set and to use Dold's fixed point transfer [10] to define τ_ε . We outline the proof for the reader's convenience.

Define $\gamma : H_g^1(M) \rightarrow \mathbb{R}$ by

$$\gamma(u) = \begin{cases} \frac{|u|_{p,\varepsilon}^p}{\|u\|_\varepsilon^2} - 1 & \text{if } u \neq 0, \\ -1 & \text{if } u = 0. \end{cases}$$

This function is continuous and satisfies

$$\gamma(u^+) = 0 = \gamma(u^-) \quad \text{if and only if } u \in \mathcal{E}_\varepsilon.$$

For $u \in H_g^1(M)$ we denote by $e(u) \in [0, \infty]$ the entrance time of u into the set \mathcal{D}_ε . That is,

$$e(u) := \inf\{t \in (0, \infty] : \varphi_\varepsilon(t, u) \in \mathcal{D}_\varepsilon\},$$

where φ_ε is the negative gradient flow of J_ε . Since \mathcal{D}_ε is strictly positively invariant, the map $e : H_g^1(M) \rightarrow [0, \infty]$ is continuous. Moreover, $e(u) = \infty$ if and only if $u \in \mathcal{Z}_\varepsilon$. Consider the retraction

$$\varrho : H_g^1(M) \setminus \mathcal{Z}_\varepsilon \rightarrow \mathcal{D}_\varepsilon, \quad \varrho(u) = \varphi_\varepsilon(e(u), u),$$

and define $\psi : H_g^1(M) \rightarrow \mathbb{R}^2$ by

$$\psi(u) = \begin{cases} 0 & \text{if } u \in \mathcal{Z}_\varepsilon, \\ \frac{1}{1+e(u)^p} (\gamma(\varrho(u)^+), \gamma(\varrho(u)^-)) & \text{if } u \in H_g^1(M) \setminus \mathcal{Z}_\varepsilon. \end{cases}$$

ψ is continuous and, since $\mathcal{D}_\varepsilon \cap \mathcal{E}_\varepsilon = \emptyset$, it satisfies

$$(3.15) \quad \psi(u) = 0 \text{ if and only if } u \in \mathcal{Z}_\varepsilon.$$

We fix $\kappa > 1$ such that

$$(3.16) \quad J_\varepsilon(\kappa(\lambda_{\varepsilon}(x, y)^+ + \mu_{\varepsilon}(x, y)^-)) \leq 0 \quad \text{if } (x, y) \in F_\varepsilon(M) \text{ and } \max\{\lambda, \mu\} \geq 1,$$

For $(x, y) \in F_\varepsilon(M)$ we define

$$g_{x,y} : [0, 1] \times [0, 1] \rightarrow \mathbb{R}^2, \quad g_{x,y}(\lambda, \mu) = \psi(\kappa(\lambda_{\varepsilon}(x, y)^+ + \mu_{\varepsilon}(x, y)^-)).$$

It follows from (3.15) that

$$(3.17) \quad \kappa(\lambda_{\varepsilon}(x, y)^+ + \mu_{\varepsilon}(x, y)^-) \in \mathcal{Z}_\varepsilon \cap J_\varepsilon^{d_\varepsilon + \delta_0} \quad \text{if } g_{x,y}(\lambda, \mu) = 0$$

and, using (3.16), we also conclude that $\kappa(\lambda_{\varepsilon}(x, y)^+ + \mu_{\varepsilon}(x, y)^-) \in \mathcal{D}_\varepsilon$ if $(\lambda, \mu) \in \partial([0, 1]^2)$. Therefore,

$$g_{x,y}(\lambda, \mu) = ((\kappa\lambda)^{p-2} - 1, (\kappa\mu)^{p-2} - 1) \quad \text{if } (\lambda, \mu) \in \partial([0, 1]^2).$$

Next we define

$$f : F_\varepsilon(M) \times [0, 1]^2 \rightarrow F_\varepsilon(M) \times \mathbb{R}^2, \quad f(x, y, \lambda, \mu) = (x, y, (\lambda, \mu) - g_{x,y}(\lambda, \mu)).$$

This map is equivariant with respect to the $\mathbb{Z}/2$ -action given by

$$-1 \cdot (x, y, \lambda, \mu) := (y, x, \mu, \lambda).$$

The projection $\pi : F_\varepsilon(M) \times \mathbb{R}^2 \rightarrow F_\varepsilon(M)$ is also equivariant and, since the action is free, the induced map between the $\mathbb{Z}/2$ -orbit spaces is a vector bundle. We denote by $\text{Fix}(\widehat{f})$ the set of fixed points of \widehat{f} . Following the proof of Lemma 5.4 in [1] one shows that they have the following properties:

- (i) $\widehat{\pi} \circ \widehat{f} = \widehat{\pi}$,
- (ii) $\text{Fix}(\widehat{f}) \subset (F_\varepsilon(M) \times (0, 1)^2) / (\mathbb{Z}/2)$,
- (iii) The fixed point transfer $\tau_{\widehat{f}} : H^*(\text{Fix}(\widehat{f})) \rightarrow H^*(C_\varepsilon(M))$ satisfies $\tau_{\widehat{f}} \circ \widehat{\pi}^* = \text{id}$.

It follows from (3.17) that the map

$$\iota : \text{Fix}(f) \rightarrow \mathcal{Z}_\varepsilon \cap J_\varepsilon^{d_\varepsilon + \delta_0}, \quad \iota(x, y, \lambda, \mu) = \kappa(\lambda \iota_\varepsilon(x, y)^+ + \mu \iota_\varepsilon(x, y)^-)$$

is well defined. It is equivariant and satisfies $\theta_\varepsilon \circ \iota = i \circ \pi|_{\text{Fix}(f)}$, where $i : F_\varepsilon(M) \hookrightarrow F(M)$ is the inclusion. We define τ_ε to be the composition

$$\tau_\varepsilon : \check{\mathcal{H}}((\mathcal{Z}_\varepsilon \cap J_\varepsilon^{d_\varepsilon + \delta_0}) / (\mathbb{Z}/2)) \xrightarrow{\widehat{\iota}^*} \check{\mathcal{H}}(\text{Fix}(\widehat{f})) \xrightarrow{\tau_{\widehat{f}}} \check{\mathcal{H}}(C_\varepsilon(M)).$$

Using property (iii) we obtain

$$\tau_\varepsilon \circ \widehat{\theta}_\varepsilon^* = \tau_{\widehat{f}} \circ \widehat{\iota}^* \circ \widehat{\theta}_\varepsilon^* = \tau_{\widehat{f}} \circ \widehat{\pi}^* \circ \widehat{i}^* = \widehat{i}^*,$$

as claimed. \square

Proof of Theorem 3.1. Fix $\delta_0 \in (0, \kappa_0)$ and $\varepsilon_0 > 0$ as above and such that the inclusion $F_\varepsilon(M) \hookrightarrow F(M)$ is a homotopy equivalence for all $\varepsilon \in (0, \varepsilon_0)$. Arguing as in section 5.2 of [1] we have that J_ε has at least $\text{cupl}[(\mathcal{Z}_\varepsilon \cap J_\varepsilon^{d_\varepsilon + \delta_0}) / (\mathbb{Z}/2)]$ sign changing solutions u_ε with $J_\varepsilon(u_\varepsilon) \leq d_\varepsilon + \delta_0$. Proposition 3.5 yields

$$\text{cupl}[(\mathcal{Z}_\varepsilon \cap J_\varepsilon^{d_\varepsilon + \delta_0}) / (\mathbb{Z}/2)] \geq \text{cupl}C(M),$$

as claimed. \square

4. THE SHAPE OF LOW ENERGY NODAL SOLUTIONS

We shall prove the following result.

Theorem 4.1. *There exists $\varepsilon_0 > 0$ such that for every sign changing solution u_ε to problem (1.1) with $\varepsilon \in (0, \varepsilon_0)$ and $J_\varepsilon(u_\varepsilon) \leq d_\varepsilon + \kappa_0$ the following statements hold true:*

- (a) u_ε has a unique local maximum point Q_ε and a unique local minimum point q_ε on M .
- (b) For any fixed $T > 0$,

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \|u_\varepsilon(\exp_{Q_\varepsilon}(\varepsilon z)) - U(z)\|_{C^2(B(0, T))} &= 0, \\ \lim_{\varepsilon \rightarrow 0} \|u_\varepsilon(\exp_{q_\varepsilon}(\varepsilon z)) + U(z)\|_{C^2(B(0, T))} &= 0. \end{aligned}$$

(c) Moreover,

$$\begin{aligned} \sup_{\xi \in M \setminus B_g(Q_\varepsilon, \varepsilon T)} u_\varepsilon(\xi) &< ce^{-\mu T} + \sigma_1(\varepsilon), \\ \inf_{\xi \in M \setminus B_g(q_\varepsilon, \varepsilon T)} u_\varepsilon(\xi) &> -ce^{-\mu T} + \sigma_2(\varepsilon), \end{aligned}$$

for some positive constants c, μ and for some functions σ_1, σ_2 which go to zero as ε goes to zero.

(d) The function Φ_ε given by

$$u_\varepsilon = W_{\varepsilon, Q_\varepsilon} - W_{\varepsilon, q_\varepsilon} + \Phi_\varepsilon$$

is such that $\|\Phi_\varepsilon\|_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

We split the proof into several lemmas. Note that, if u_ε is a sign changing solution to problem (1.1) then $u_\varepsilon \in \mathcal{C}^2(M)$. Hence, it has a maximum point Q_ε and a minimum point q_ε on M . Moreover, $u_\varepsilon(Q_\varepsilon) > 0$ and $u_\varepsilon(q_\varepsilon) < 0$. The following estimates hold true.

Lemma 4.2. *If u_ε is a sign changing solution to problem (1.1) and Q_ε is a maximum point and q_ε is a minimum point of u_ε on M , then*

$$u_\varepsilon(Q_\varepsilon) \geq 1 \quad \text{and} \quad u_\varepsilon(q_\varepsilon) \leq -1.$$

Proof. Expressing u_ε in local normal coordinates around the point Q_ε we get

$$\tilde{u}_\varepsilon(z) := u_\varepsilon(\exp_{Q_\varepsilon}(z)) \quad \text{for } |z| < R, \ z \in \mathbb{R}^n.$$

Recall that in these coordinates we have

$$(4.1) \quad \Delta_g v = \frac{1}{\sqrt{|g|}} \sum_{ij} \frac{\partial}{\partial z_i} \left(g^{ij}(z) \sqrt{|g|} \frac{\partial v}{\partial z_j} \right) \quad \text{and} \quad g^{ij}(0) = \delta_{ij},$$

where $|g(z)| := \det(g_{ij}(z))$ and $(g^{ij}(z))$ is the inverse matrix of $(g_{ij}(z))$. Hence,

$$(4.2) \quad -\varepsilon^2 \sum_{i=1}^n \frac{\partial^2}{\partial y_i^2} \tilde{u}_\varepsilon(0) + \tilde{u}_\varepsilon(0) = (\tilde{u}_\varepsilon(0))^{p-1}.$$

Since 0 a maximum point of \tilde{u}_ε we get from (4.2) that $1 \leq \tilde{u}_\varepsilon(0) = u_\varepsilon(Q_\varepsilon)$. This proves the first inequality. The proof of the second one is similar. \square

In the following two lemmas we assume that $\varepsilon_k \in (0, 1)$ is such that ε_k converges to 0 and that u_{ε_k} is a sign changing solution to problem (1.1) with $\varepsilon := \varepsilon_k$ which satisfies $J_{\varepsilon_k}(u_{\varepsilon_k}) \leq d_{\varepsilon_k} + \kappa_0$, where κ_0 is as in Theorem 2.7. Note that

$$\frac{2p}{p-2} d_{\varepsilon_k} \leq \|u_{\varepsilon_k}\|_{\varepsilon_k}^2 = \frac{2p}{p-2} J_{\varepsilon_k}(u_{\varepsilon_k}) \leq \frac{2p}{p-2} (d_{\varepsilon_k} + \kappa_0).$$

So, by Proposition 2.6, there are constants c_1, c_2 such that, for k large enough,

$$(4.3) \quad 0 < c_1 \leq \|u_{\varepsilon_k}\|_{\varepsilon_k}^2 \leq c_2 < \infty.$$

Let $Q_k := Q_{\varepsilon_k}$ be a maximum point and $q_k := q_{\varepsilon_k}$ be a minimum point of u_{ε_k} on M , and set

$$\begin{aligned} w_k^1(z) &:= u_{\varepsilon_k}(\exp_{Q_k}(\varepsilon_k z)) \chi(\varepsilon_k |z|) & \text{for } z \in \mathbb{R}^n, \\ w_k^2(z) &:= u_{\varepsilon_k}(\exp_{q_k}(\varepsilon_k z)) \chi(\varepsilon_k |z|) & \text{for } z \in \mathbb{R}^n, \\ \tilde{w}_k^1(z) &:= u_{\varepsilon_k}(\exp_{Q_k}(\varepsilon_k z)) & \text{for } z \in B\left(0, \frac{R}{\varepsilon_k}\right) \subset \mathbb{R}^n, \\ \tilde{w}_k^2(z) &:= u_{\varepsilon_k}(\exp_{q_k}(\varepsilon_k z)) & \text{for } z \in B\left(0, \frac{R}{\varepsilon_k}\right) \subset \mathbb{R}^n, \end{aligned}$$

where $\chi : \mathbb{R}^+ \rightarrow [0, 1]$ is a \mathcal{C}^∞ cut-off function such that $\chi(t) \equiv 1$ if $0 \leq t \leq R/2$ and $\chi(t) \equiv 0$ if $R \leq t$. Note that $\tilde{w}_k^1(z) = w_k^1(z)$ and $\tilde{w}_k^2(z) = w_k^2(z)$ for $|z| < \frac{R}{2\varepsilon_k}$.

Lemma 4.3. *There exist $w^1, w^2 \in H^1(\mathbb{R}^n)$ such that, after passing to a subsequence, $w_k^i \rightharpoonup w^i$ weakly in $H^1(\mathbb{R}^n)$, and $w_k^i \rightarrow w^i$ strongly in $L_{loc}^p(\mathbb{R}^n)$ and in $\mathcal{C}_{loc}^2(\mathbb{R}^n)$ for $i = 1, 2$. The functions w^1 and w^2 are nontrivial solutions of the equation*

$$(4.4) \quad -\Delta w + w = |w|^{p-2}w$$

such that 0 is a maximum point of w^1 and a minimum point of w^2 .

Proof. Arguing as in the proof of Lemma 5.6 in [2] one shows that $\|w_k^i\|_{H^1(\mathbb{R}^n)} \leq c \|u_{\varepsilon_k}\|_{\varepsilon_k}$ for some positive constant c independent of k . It follows from (4.3) that (w_k^i) is bounded in $H^1(\mathbb{R}^n)$. Hence, there exists $w^i \in H^1(\mathbb{R}^n)$ such that a subsequence of (w_k^i) converges to w^i weakly in $H^1(\mathbb{R}^n)$ and strongly in $L_{loc}^p(\mathbb{R}^n)$. The fact that w^i solves (4.4) can be proved as in Lemma 5.7 of [2]. On the other hand, for $|z| < R/\varepsilon_k$ the function \tilde{w}_k^i satisfies the following equation

$$-\frac{1}{\sqrt{|g(\varepsilon_k z)|}} \sum_{ij} \frac{\partial}{\partial z_i} \left(g^{ij}(\varepsilon_k z) \sqrt{|g(\varepsilon_k z)|} \frac{\partial \tilde{w}_k^i}{\partial z_j} \right) + \tilde{w}_k^i = |\tilde{w}_k^i|^{p-2} \tilde{w}_k^i.$$

Arguing as above, we have that the sequence (\tilde{w}_k^i) is bounded in $H^1(B(0, R/\varepsilon_k))$. By the Sobolev embedding theorem and interior Schauder estimates in $B(0, \frac{R}{\varepsilon_k})$ we get that (\tilde{w}_k^i) is bounded in $\mathcal{C}^{2,\alpha}(B(0, R/\varepsilon_k))$ for some $\alpha \in (0, 1)$. Then, up to subsequence we have

$$\tilde{w}_k^i \rightarrow \tilde{w}^i \quad \text{in } \mathcal{C}_{loc}^2(\mathbb{R}^n).$$

Clearly, $\tilde{w}^i = w^i \in H^1(\mathbb{R}^n) \cap \mathcal{C}^2(\mathbb{R}^n)$. Since $w_k^1(0) = u_{\varepsilon_k}(Q_k) \geq 1$ for any k , we have that $w^1 \neq 0$. Similarly for w^2 . \square

Lemma 4.4. *The functions w^1 and w^2 do not change sign. Moreover,*

$$w^1 = -w^2 = U.$$

Proof. By Theorem 2.7, in order to prove that w^i does not change sign it suffices to show that

$$(4.5) \quad J_\infty(w^i) \leq 2c_\infty + \kappa_0.$$

Since $|g(0)| = 1$ we have that, for any $\delta > 0$, there exists $\rho > 0$ such that $|g(y)|^{-1/2} < 1 + \delta$ for $|y| < \rho$. Using this fact we obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |w^1(z)|^p dz &\leq \liminf_{k \rightarrow \infty} \int_{\mathbb{R}^n} |w_k^1(z)|^p dz \\ &= \liminf_{k \rightarrow \infty} \frac{1}{\varepsilon_k^n} \int_{|y| < R\varepsilon_k} \left| u_k(\exp_{Q_k}(y)) \chi\left(\frac{|y|}{\varepsilon_k}\right) \right|^p \frac{|g(y)|^{1/2}}{|g(y)|^{1/2}} dy \\ &\leq (1 + \delta) \liminf_{k \rightarrow \infty} \frac{1}{\varepsilon_k^n} \int_{B_g(Q_k, R\varepsilon_k)} |u_{\varepsilon_k}(x)|^p d\mu_g \\ &\leq (1 + \delta) \liminf_{k \rightarrow \infty} \frac{1}{\varepsilon_k^n} \int_M |u_{\varepsilon_k}(x)|^p d\mu_g. \end{aligned}$$

Multiplying by $\frac{p-2}{2p}$ and using Proposition 2.6 we conclude that

$$J_\infty(w^1) \leq (1 + \delta) \liminf_{k \rightarrow \infty} J_{\varepsilon_k}(u_{\varepsilon_k}) \leq (1 + \delta) \liminf_{k \rightarrow \infty} (d_{\varepsilon_k} + \kappa_0) = (1 + \delta) (2c_\infty + \kappa_0).$$

Since $\delta > 0$ is arbitrary, this yields inequality (4.5). We conclude that w^1 is a nontrivial solution to (4.4) which does not change sign. Since 0 is a maximum point of w^1 it follows that $w^1 = U$. The statements for w^2 are proved similarly. \square

Lemmas 4.3 and 4.4 together imply that

$$(4.6) \quad \tilde{w}_k^1 \rightarrow U \quad \text{and} \quad \tilde{w}_k^2 \rightarrow -U \quad \text{in } C_{loc}^2(\mathbb{R}^n).$$

Lemma 4.5. *If u_ε is a sign changing solution to problem (1.1) which satisfies $J_\varepsilon(u_\varepsilon) \leq d_\varepsilon + \kappa_0$ then, for ε small enough, u_ε has a unique local maximum point Q_ε and a unique local minimum point q_ε on M .*

Proof. Arguing by contradiction we assume there is a sequence (ε_k) which goes to zero and, for each k , a sign changing solution u_{ε_k} of problem (1.1) which satisfies $J_{\varepsilon_k}(u_{\varepsilon_k}) \leq d_{\varepsilon_k} + \kappa_0$ and has three local extrema q_k^1, q_k^2 and q_k^3 . As before, we set

$$\tilde{w}_k^i(z) := u_{\varepsilon_k}(\exp_{q_k^i}(\varepsilon_k z)) \quad \text{for } z \in B\left(0, \frac{R}{\varepsilon_k}\right), \quad i = 1, 2, 3.$$

Fix $\kappa_1 \in (\kappa_0, c_\infty)$ and choose $T > 0$ such that

$$(4.7) \quad \frac{p-2}{2p} \int_{B(0,T)} |U(z)|^p dz > \frac{2c_\infty + \kappa_1}{3}.$$

It follows from (4.6) that, for k large enough, \tilde{w}_k^i has a unique local extremum point at 0 in $B(0, 2T)$. Hence, u_{ε_k} has a unique local extremum point at q_k^i in $B_g(q_k^i, 2\varepsilon_k T)$, for each $i = 1, 2, 3$. In particular, $B_g(q_k^i, \varepsilon_k T) \cap B_g(q_k^j, \varepsilon_k T) = \emptyset$ if $i \neq j$. On the other hand, since $|g(0)| = 1$, for any $\delta > 0$ and k large enough we have

$$(4.8) \quad c_0 := \frac{2c_\infty + \kappa_0}{2c_\infty + \kappa_1} < |g(\varepsilon_k z)|^{\frac{1}{2}} \quad \text{for } |z| < T \text{ and } k \text{ large enough.}$$

Therefore, for all sufficiently large k we obtain

$$\begin{aligned}
 \frac{1}{\varepsilon_k^n} \int_M |u_{\varepsilon_k}|^p d\mu_g &\geq \frac{1}{\varepsilon_k^n} \sum_{i=1}^3 \int_{B_g(q_k^i, \varepsilon_k T)} |u_{\varepsilon_k}|^p d\mu_g \\
 &= \sum_{i=1}^3 \int_{B(0,T)} |\tilde{w}_k^i(z)|^p |g(\varepsilon_k z)|^{\frac{1}{2}} dz \\
 (4.9) \qquad &\geq c_0 \sum_{i=1}^3 \int_{B(0,T)} |\tilde{w}_k^i(z)|^p dz.
 \end{aligned}$$

Multiplying by $\frac{p-2}{2p}$ we conclude that

$$d_{\varepsilon_k} + \kappa_0 \geq J_{\varepsilon_k}(u_{\varepsilon_k}) \geq c_0 \sum_{i=1}^3 \frac{p-2}{2p} \int_{B(0,T)} |\tilde{w}_k^i(z)|^p dz.$$

Passing to the limit as $k \rightarrow \infty$, Proposition 2.6 and Lemmas 4.3 and 4.4 yield

$$\begin{aligned}
 2c_\infty + \kappa_0 &= \lim_{k \rightarrow \infty} d_{\varepsilon_k} + \kappa_0 \geq c_0 \sum_{i=1}^3 \frac{p-2}{2p} \lim_{k \rightarrow \infty} \int_{B(0,T)} |\tilde{w}_k^i(z)|^p dz \\
 &= 3c_0 \frac{p-2}{2p} \int_{B(0,T)} |U(z)|^p dz > c_0(2c_\infty + \kappa_1) = 2c_\infty + \kappa_0.
 \end{aligned}$$

This is a contradiction. \square

Lemma 4.6. *Fix $T > 0$. If u_ε is a sign changing solution to problem (1.1) which satisfies $J_\varepsilon(u_\varepsilon) \leq d_\varepsilon + \kappa_0$ then, for ε small enough, there are constants c, μ and functions σ_1, σ_2 such that*

$$\begin{aligned}
 \sup_{\xi \in M \setminus B_g(Q_\varepsilon, \varepsilon T)} u_\varepsilon(\xi) &< ce^{-\mu T} + \sigma_1(\varepsilon), \\
 \inf_{\xi \in M \setminus B_g(Q_\varepsilon, \varepsilon T)} u_\varepsilon(\xi) &> -ce^{-\mu T} + \sigma_2(\varepsilon),
 \end{aligned}$$

and $\lim_{\varepsilon \rightarrow 0} \sigma_i(\varepsilon) = 0$ for $i = 1, 2$.

Proof. Since for ε small enough u_ε has a unique local maximum, the supremum of u_ε on $M \setminus B_g(Q_\varepsilon, \varepsilon T)$ is attained at a point of the boundary $\partial B_g(Q_\varepsilon, \varepsilon T)$. We consider the function

$$\tilde{w}_\varepsilon^1(z) = u_\varepsilon(\exp_{Q_\varepsilon}(\varepsilon z)) \quad \text{for } |z| \leq T.$$

Then, using the decay (2.2) of U , for some constants $c, \mu > 0$ we have

$$\begin{aligned}
 \sup_{\xi \in M \setminus B_g(Q_\varepsilon, \varepsilon T)} u_\varepsilon(\xi) &\leq \sup_{|z|=T} |\tilde{w}_\varepsilon^1(z)| \leq \sup_{|z|=T} U(z) + \sup_{|z|=T} |\tilde{w}_\varepsilon^1(z) - U(z)| \\
 &\leq ce^{-\mu T} + \sup_{|z|=T} |\tilde{w}_\varepsilon^1(z) - U(z)|.
 \end{aligned}$$

By (4.6) we have that

$$\sigma_1(\varepsilon) := \sup_{|z| \leq T} |\tilde{w}_\varepsilon^1(z) - U(z)| \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0.$$

This proves the first inequality. Analogously for the other one. \square

Lemma 4.7. *If u_ε is a sign changing solution to problem (1.1) which satisfies $J_\varepsilon(u_\varepsilon) \leq d_\varepsilon + \kappa_0$ then*

$$(4.10) \quad \lim_{\varepsilon \rightarrow 0} J_\varepsilon(u_\varepsilon) = 2c_\infty,$$

and the function Φ_ε given by

$$u_\varepsilon = W_{\varepsilon, Q_\varepsilon} - W_{\varepsilon, q_\varepsilon} + \Phi_\varepsilon$$

satisfies that $\|\Phi_\varepsilon\|_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$.

Proof. For fixed $T > 0$ we set $B_{\varepsilon T}^1 := B_g(Q_\varepsilon, \varepsilon T)$, $B_{\varepsilon T}^2 := B_g(q_\varepsilon, \varepsilon T)$ and $A_{\varepsilon T} := M \setminus (B_{\varepsilon T}^1 \cup B_{\varepsilon T}^2)$. Recall that for ε small enough we have $2T\varepsilon < d_g(Q_\varepsilon, q_\varepsilon)$. For $D \subset M$ set

$$\|v\|_{\varepsilon, D}^2 := \frac{1}{\varepsilon^n} \int_D (\varepsilon^2 |\nabla_g v|^2 + v^2) d\mu_g, \quad |v|_{p, \varepsilon, D}^p := \frac{1}{\varepsilon^n} \int_D |v|^p d\mu_g.$$

Then, for $i = 1, 2$ we have

$$\begin{aligned} \|u_\varepsilon\|_{\varepsilon, B_{\varepsilon T}^i}^2 &= \frac{1}{\varepsilon^n} \int_{B_{\varepsilon T}^i} (\varepsilon^2 |\nabla_g u_\varepsilon|^2 + u_\varepsilon^2) d\mu_g \\ &= \int_{|z| < T} \left[\sum_{jm} g^{jm}(\varepsilon z) \frac{\partial}{\partial z_j} \tilde{w}_\varepsilon^i(z) \frac{\partial}{\partial z_m} \tilde{w}_\varepsilon^i(z) + (\tilde{w}_\varepsilon^i(z))^2 \right] |g(\varepsilon z)|^{\frac{1}{2}} dz \\ &\rightarrow \int_{|z| < T} (|\nabla U(z)|^2 + U(z)^2) dz =: c_T \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

It follows that

$$(4.11) \quad \|u_\varepsilon\|_\varepsilon^2 - \|u_\varepsilon\|_{\varepsilon, A_{\varepsilon T}}^2 \rightarrow 2c_T \quad \text{as } \varepsilon \rightarrow 0.$$

Similarly,

$$(4.12) \quad |u_\varepsilon|_{p, \varepsilon}^p - |u_\varepsilon|_{p, \varepsilon, A_{\varepsilon T}}^p \rightarrow 2c_{p, T} \quad \text{as } \varepsilon \rightarrow 0,$$

where

$$c_{p, T} := \int_{|z| < T} U(z)^p dz.$$

We write Φ_ε as

$$(4.13) \quad \begin{aligned} \Phi_\varepsilon &= (u_\varepsilon - W_{\varepsilon, Q_\varepsilon})|_{B_{\varepsilon T}^1} + (u_\varepsilon - W_{\varepsilon, q_\varepsilon})|_{B_{\varepsilon T}^2} \\ &\quad + u_\varepsilon|_{A_{\varepsilon T}} - W_{\varepsilon, Q_\varepsilon}|_{B_g(Q_\varepsilon, \varepsilon R) \setminus B_\varepsilon^1} + W_{\varepsilon, q_\varepsilon}|_{B_g(q_\varepsilon, \varepsilon R) \setminus B_\varepsilon^2}. \end{aligned}$$

Using (4.6) we obtain

$$\begin{aligned} (4.14) \quad &\|u_\varepsilon - W_{\varepsilon, Q_\varepsilon}\|_{\varepsilon, B_{\varepsilon T}^1}^2 \\ &= \int_{|z| < T} \left[\sum_{jm} g^{jm}(\varepsilon z) \frac{\partial}{\partial z_j} (\tilde{w}_\varepsilon^1(z) - U(z)\chi(\varepsilon z)) \frac{\partial}{\partial z_m} (\tilde{w}_\varepsilon^1(z) - U(z)\chi(\varepsilon z)) \right. \\ &\quad \left. + (\tilde{w}_\varepsilon^1(z) - U(z)\chi(\varepsilon z))^2 \right] |g(\varepsilon z)|^{\frac{1}{2}} dz \rightarrow 0 \quad \text{as } \varepsilon \rightarrow 0. \end{aligned}$$

Similarly, as $\varepsilon \rightarrow 0$,

$$\|u_\varepsilon - W_{\varepsilon, q_\varepsilon}\|_{\varepsilon, B_{\varepsilon T}^2}^2 \rightarrow 0, \quad |u_\varepsilon - W_{\varepsilon, Q_\varepsilon}|_{p, \varepsilon, B_{\varepsilon T}^1}^p \rightarrow 0, \quad |u_\varepsilon - W_{\varepsilon, q_\varepsilon}|_{p, \varepsilon, B_{\varepsilon T}^2}^p \rightarrow 0.$$

Moreover, there is a constant C such that, for all $\varepsilon \in (0, 1)$,

(4.15)

$$\begin{aligned} & \|W_{\varepsilon, Q_\varepsilon}\|_{\varepsilon, B_g(Q_\varepsilon, \varepsilon R) \setminus B_\varepsilon^1}^2 \\ &= \int_{T < |z| < \frac{R}{\varepsilon}} |g(\varepsilon z)|^{\frac{1}{2}} \left[\sum_{jm} g^{jm}(\varepsilon z) \frac{\partial}{\partial z_j} (U(z) \chi(\varepsilon z)) \frac{\partial}{\partial z_m} (U(z) \chi(\varepsilon z)) + (U(z) \chi(\varepsilon z))^2 \right] dz \\ &\leq C \int_{|z| > T} (|\nabla U(z)|^2 + U^2(z)) dz, \end{aligned}$$

Similarly,

$$(4.16) \quad \|W_{\varepsilon, q_\varepsilon}\|_{\varepsilon, B_g(q_\varepsilon, \varepsilon R) \setminus B_\varepsilon^2}^2, \quad |W_{\varepsilon, Q_\varepsilon}|_{p, \varepsilon, B_g(Q_\varepsilon, \varepsilon R) \setminus B_\varepsilon^1}^p, \quad |W_{\varepsilon, q_\varepsilon}|_{p, \varepsilon, B_g(Q_\varepsilon, \varepsilon R) \setminus B_\varepsilon^2}^p,$$

are bounded above by a function of T which goes to zero as $T \rightarrow \infty$.

To prove (4.10) we argue by contradiction. Assume there is a sequence $\varepsilon_k \rightarrow 0$ such that $J_{\varepsilon_k}(u_{\varepsilon_k}) \rightarrow 2c_\infty + \kappa_1$ with $\kappa_1 \in (0, \kappa_0]$. Then (4.11) and (4.12) imply that, as $k \rightarrow \infty$,

(4.17)

$$\|u_{\varepsilon_k}\|_{\varepsilon_k, A_{\varepsilon_k T}}^2 \rightarrow \frac{2p}{p-2}(2c_\infty + \kappa_1) - 2c_T, \quad |u_{\varepsilon_k}|_{p, \varepsilon_k, A_{\varepsilon_k T}}^p \rightarrow \frac{2p}{p-2}(2c_\infty + \kappa_1) - 2c_{p,T}.$$

Let $\delta \in (0, 1)$. Since $c_T \rightarrow \frac{2p}{p-2}c_\infty$, $c_{p,T} \rightarrow \frac{2p}{p-2}c_\infty$, and the right hand sides of (4.15) and of the similar inequalities for (4.16) tend to zero as $T \rightarrow \infty$, we may first choose $T > 0$ and then choose $k_0 = k_0(T) \in \mathbb{N}$ such that from (4.13), (4.14), (4.15) and (4.17) we obtain

$$\|\Phi_{\varepsilon_k}\|_{\varepsilon_k}^2 \leq (1 + \delta) \frac{2p}{p-2} \kappa_1 \quad \text{and} \quad |\Phi_{\varepsilon_k}|_{p, \varepsilon_k}^p \geq (1 - \delta) \frac{2p}{p-2} \kappa_1 \quad \forall k \geq k_0.$$

Since we are assuming that $\kappa_1 > 0$ we have that $\Phi_{\varepsilon_k} \neq 0$. Then, as in (2.4), $t_{\varepsilon_k}(\Phi_{\varepsilon_k})\Phi_{\varepsilon_k} \in \mathcal{N}_{\varepsilon_k}$ and

$$J_{\varepsilon_k}(t_{\varepsilon_k}(\Phi_{\varepsilon_k})\Phi_{\varepsilon_k}) = \frac{p-2}{2p} \left(\frac{\|\Phi_{\varepsilon_k}\|_{\varepsilon_k}^2}{|\Phi_{\varepsilon_k}|_{p, \varepsilon_k}^2} \right)^{\frac{p}{p-2}} \leq \frac{(1 + \delta)^{\frac{p}{p-2}}}{(1 - \delta)^{\frac{2}{p-2}}} \kappa_1.$$

Choosing δ small enough so that the right hand side of this inequality is smaller than c_∞ we obtain a contradiction to Lemma 2.1. This proves (4.10).

Identity (4.10) together with (4.11) implies that

$$\|u_\varepsilon\|_{\varepsilon, A_{\varepsilon T}}^2 \rightarrow 2 \left(\frac{2p}{p-2} c_\infty - c_T \right) \quad \text{as } \varepsilon \rightarrow 0.$$

Hence, for any $\eta > 0$ we may choose $T > 0$ large enough and $\varepsilon_0 = \varepsilon_0(T) > 0$ so that from (4.13), (4.14), (4.15) and (4.17) we obtain that

$$\|\Phi_\varepsilon\|_\varepsilon^2 < \eta \quad \text{for every } \varepsilon \in (0, \varepsilon_0).$$

This finishes de proof. \square

Proof of Theorem 4.1. Parts (a), (b), (c) and (d) are given by Lemma 4.5, statement (4.6) and Lemmas 4.6 and 4.7 respectively. \square

5. THE CUP-LENGTH OF CONFIGURATION SPACES

Let $\pi : TM \rightarrow M$ be the tangent bundle of M , whose fiber over x is the tangent space $T_x M$ to M at x , and let $\pi : \mathbb{S}M \rightarrow M$ be its unit-sphere bundle. The group $\mathbb{Z}/2$ acts on $\mathbb{S}M$ by multiplication on each fiber, i.e. $-1 \cdot (x, z) = (x, -z)$ for every $x \in M$ and $z \in T_x M$ with $|z| = 1$. We denote its $\mathbb{Z}/2$ -orbit space by $\mathbb{P}M$. Then, π induces a map $\hat{\pi} : \mathbb{P}M \rightarrow M$ which is a fiber bundle with fiber the real projective space $\mathbb{R}P^{n-1}$. The homomorphism $\theta : \mathcal{H}^*(\mathbb{R}P^{n-1}) \rightarrow \mathcal{H}^*(\mathbb{P}M)$ which sends the generator $\omega \in \mathcal{H}^1(\mathbb{R}P^{n-1})$ to the first Stiefel-Whitney class $\hat{\omega} \in \mathcal{H}^1(\mathbb{P}M)$ of the bundle $\mathbb{S}M \rightarrow \mathbb{P}M$ is a cohomology extension of the fiber, so the Leray-Hirsch theorem [23] provides an isomorphism

$$(5.1) \quad \mathcal{H}^*(M) \otimes \mathcal{H}^*(\mathbb{R}P^{n-1}) \cong \mathcal{H}^*(\mathbb{P}M)$$

given by $\zeta \otimes \omega \mapsto \hat{\pi}^*(\zeta) \smile \hat{\omega}$.

If V is a tubular neighborhood of M in \mathbb{R}^N , we define a map $\alpha : \mathbb{S}M \rightarrow F(V)$ by

$$\alpha(x, z) := (\exp_x(\frac{R}{2}z), \exp_x(-\frac{R}{2}z)).$$

The action of $\mathbb{Z}/2$ on $F(V)$ is given by $-1 \cdot (x, y) = (y, x)$. Therefore, α is $\mathbb{Z}/2$ -equivariant, i.e. $\alpha(x, -z) = -1 \cdot \alpha(x, z)$, and it induces a map between the $\mathbb{Z}/2$ -orbit spaces, which we denote by

$$\hat{\alpha} : \mathbb{P}M \rightarrow C(V).$$

The cup-length of a map $f : X \rightarrow Y$ is the smallest integer $k \geq 1$ such that $f^*(\zeta_1 \smile \cdots \smile \zeta_k) = 0$ for any k cohomology classes $\zeta_1, \dots, \zeta_k \in \tilde{\mathcal{H}}^*(Y)$. It is denoted $\text{cupl}(f)$. If f is an inclusion $X \hookrightarrow Y$ we write $\text{cupl}_Y X := \text{cupl}(f)$. It is easy to see that $\text{cupl}(g \circ f) \leq \min\{\text{cupl}(f), \text{cupl}(g)\}$, cf. [5]. Since the image of α is contained in $C(M)$, we conclude that

$$(5.2) \quad \text{cupl}(\hat{\alpha}) \leq \text{cupl}_{C(V)} C(M) \leq \text{cupl} C(M).$$

To prove Theorem 1.2 we need the following lemma. Its proof uses the Leray-Serre spectral sequence, which is treated for example in [17].

Lemma 5.1. *Let V be a tubular neighborhood of M in \mathbb{R}^N . Assume that $\mathcal{H}^i(M) = 0$ for all $0 < i < m$. Then, given a cohomology class $\zeta \in \mathcal{H}^m(M)$, there exists $\hat{\zeta} \in \mathcal{H}^m(C(V))$ such that*

$$\hat{\alpha}^*(\hat{\zeta}) = \hat{\pi}^*(\zeta).$$

Proof. Consider the diagram

$$\begin{array}{ccc} \mathbb{P}M & \xrightarrow{\hat{\alpha}} & C(V) \\ \phi \uparrow & & \uparrow \psi \\ \mathbb{S}M & \xrightarrow{\alpha} & F(V) \\ \pi \downarrow & & \downarrow \pi_1 \\ M & \xrightarrow{i} & V \end{array}$$

where π_1 is the projection onto the first factor, and ϕ and ψ are the obvious projections. Thus, $\hat{\pi} \circ \phi = \pi$. This diagram commutes up to homotopy. Since M is a strong deformation retract of V , the inclusion i induces an isomorphism in cohomology. Hence, there exists $\bar{\zeta} \in \mathcal{H}^m(F(V))$ such that $\alpha^*(\bar{\zeta}) = \pi^*(\zeta) \in \mathcal{H}^m(\mathbb{S}M)$. Next, we will show that $\bar{\zeta} = \psi^*(\tilde{\zeta})$ for some $\tilde{\zeta} \in \mathcal{H}^m(C(V))$.

From the Thom-Gysin sequence of the sphere bundle $\pi : \mathbb{S}M \rightarrow M$ we obtain that $\pi^* : \mathcal{H}^i(M) \rightarrow \mathcal{H}^i(\mathbb{S}M)$ is an isomorphism for all $i < m + n - 1$ and a monomorphism for $i = m + n - 1$. On the other hand, setting $D_\varepsilon V := \{(x, y) \in V \times V : |x - y| \leq \varepsilon\}$, and using excision, homotopy invariance and the Thom isomorphism we obtain, for ε small enough,

$$\mathcal{H}^i(V \times V, F(V)) \cong \mathcal{H}^i(D_\varepsilon V, D_\varepsilon V \setminus D_0 V) \cong \tilde{\mathcal{H}}^{i-N}(V) \cong \tilde{\mathcal{H}}^{i-N}(M).$$

From the exact cohomology sequence of the pair $(V \times V, F(V))$ we deduce that

$$(5.3) \quad \mathcal{H}^i(F(V)) \cong \mathcal{H}^i(V \times V) \cong \mathcal{H}^i(M \times M) \quad \text{for all } i < m + N - 1.$$

We consider the Leray-Serre spectral sequences of the Borel fibrations (for the group $G = \mathbb{Z}/2$)

$$F(V) \times_{\mathbb{Z}/2} \mathbb{S}^\infty \rightarrow \mathbb{R}P^\infty, \quad \mathbb{S}M \times_{\mathbb{Z}/2} \mathbb{S}^\infty \rightarrow \mathbb{R}P^\infty.$$

Since $\mathbb{Z}/2$ acts freely on $F(V)$ and on $\mathbb{S}M$, the projections $F(V) \times \mathbb{S}^\infty \rightarrow F(V)$ and $\mathbb{S}M \times \mathbb{S}^\infty \rightarrow \mathbb{S}M$ induce homotopy equivalences between the $\mathbb{Z}/2$ -orbit spaces

$$F(V) \times_{\mathbb{Z}/2} \mathbb{S}^\infty \simeq C(V), \quad \mathbb{S}M \times_{\mathbb{Z}/2} \mathbb{S}^\infty \simeq \mathbb{P}M.$$

The map $\alpha : \mathbb{S}M \rightarrow F(V)$ induces a map of spectral sequences

$$\alpha^* : E_r^{p,q} \rightarrow \tilde{E}_r^{p,q}.$$

Their E_2 -terms are

$$E_2^{p,q} = \mathcal{H}^p(\mathbb{R}P^\infty; \mathcal{H}^q(F(V))), \quad \tilde{E}_2^{p,q} = \mathcal{H}^p(\mathbb{R}P^\infty; \mathcal{H}^q(\mathbb{S}M)).$$

Our assumptions on $\mathcal{H}^*(M)$ together with (5.3) imply that $\mathcal{H}^q(F(V)) = 0$ if $0 < q < m$. Therefore,

$$\begin{aligned} \mathcal{H}^m(F(V)) &\cong E_2^{0,m} = \dots = E_{m+1}^{0,m}, \\ \mathcal{H}^{m+1}(\mathbb{R}P^\infty) &\cong E_2^{m+1,0} = \dots = E_{m+1}^{m+1,0}. \end{aligned}$$

Since $\pi^*(\zeta)$ is a permanent cycle and $\alpha^*(\bar{\zeta}) = \pi^*(\zeta)$, we have that

$$\alpha^* d_{m+1}(\bar{\zeta}) = \tilde{d}_{m+1}(\pi^*(\zeta)) = 0.$$

But α^* is the identity on $\mathcal{H}^{m+1}(\mathbb{R}P^\infty) \cong E_2^{m+1,0} = \tilde{E}_2^{m+1,0}$. Hence $d_{m+1}(\bar{\zeta}) = 0$ and, therefore, $\bar{\zeta}$ is a permanent cycle too. Thus, there exists $\tilde{\zeta} \in \mathcal{H}^m(C(V))$ such that $\psi^*(\tilde{\zeta}) = \bar{\zeta}$.

Our assumptions on $\mathcal{H}^*(M)$, together with (5.1), imply that $\mathcal{H}^m(\mathbb{P}M) \cong \mathcal{H}^m(M) \oplus \mathcal{H}^m(\mathbb{R}P^{n-1})$. Since $\phi^* \hat{\alpha}^*(\tilde{\zeta}) = \phi^* \pi^*(\zeta)$ we conclude that $\hat{\alpha}^*(\tilde{\zeta})$ is either $\hat{\pi}^*(\zeta)$ or $\hat{\pi}^*(\zeta) + \tilde{\omega}^m$. In the first case we set $\hat{\zeta} := \tilde{\zeta}$ and in the second case we set $\hat{\zeta} := \tilde{\zeta} - \tilde{\omega}^m$, where $\tilde{\omega} \in \mathcal{H}^1(C(M))$ is the first Stiefel-Whitney class of the bundle $F(V) \rightarrow C(V)$. Since $\hat{\alpha}^*(\tilde{\omega}) = \tilde{\omega}$, we conclude that $\hat{\alpha}^*(\hat{\zeta}) = \hat{\pi}^*(\zeta)$, as claimed. \square

Proof of Theorem 1.2. By (5.2) it suffices to show that $\text{cupl}(\hat{\alpha}) \geq k + n$. Let $\zeta_1, \dots, \zeta_k \in \mathcal{H}^m(M)$ be such that $\zeta_1 \smile \dots \smile \zeta_k \neq 0$. Then (5.1) yields $\hat{\pi}^*(\zeta_1 \smile \dots \smile \zeta_k) \smile \tilde{\omega}^{n-1} \neq 0$. By Lemma 5.1 there exist $\hat{\zeta}_1, \dots, \hat{\zeta}_k \in \mathcal{H}^m(C(V))$ such that $\hat{\alpha}^*(\hat{\zeta}_i) = \hat{\pi}^*(\zeta_i)$. Therefore,

$$\hat{\alpha}^*(\hat{\zeta}_1 \smile \dots \smile \hat{\zeta}_k \smile \tilde{\omega}^{n-1}) = \hat{\pi}^*(\zeta_1 \smile \dots \smile \zeta_k) \smile \tilde{\omega}^{n-1} \neq 0.$$

It follows that $\text{cupl}(\hat{\alpha}) \geq k + n$. \square

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